Oppgaver i 5.4 Integrasjonsmetoder og 5.5 Volumberegninger.

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a) Delvis:
$\int x^2 \ln x dx = \frac{x^3}{3}\ln x - \frac{1}{3}\int x^3 \frac{1}{x} dx = \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2 dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C = \frac{x^3}{9}(3\ln x - 1) + C$

b) Variabelskifte: $u = \ln x \Leftrightarrow u' = \frac{1}{x} \Leftrightarrow dx = xdu$

$\int \frac{\ln x}{x} dx = \int \frac{u}{x} xdu = \int udu = \frac{u^2}{2} + C = \frac{\ln^2 x}{2} + C$

c) Delvis: $I = \int x(\ln x)^2 = \frac{x^2}{2}\ln^2 x - \frac{1}{2}\int x^2 (2\ln x)\frac{1}{x} dx = \frac{x^2}{2}\ln^2 x - \int x \ln x dx$

Mellomregning: $\int x \ln x dx = \frac{x^2}{2}\ln x - \frac{1}{2}\int x^2 \frac{1}{x} dx = \frac{x^2}{2}\ln x - \frac{1}{2}\int x dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$

): $I = \frac{x^2}{2}\ln^2 x - (\frac{x^2}{2}\ln x - \frac{x^2}{4} + C) = \frac{x^2}{4}(2\ln^2 x - 2\ln x + 1) + D$

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a) Polynomdivisjon: $\int \frac{x^2-1}{x} dx = \int (x - \frac{1}{x}) dx = \frac{x^2}{2} - \ln|x| + C$

b) $\int \frac{x^3+2x^2-1}{x^2} dx = \int (x + 2 - x^{-2}) dx = \frac{x^2}{2} + 2x + x^{-1} + C$

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d) Delbrøk: $I = \int \frac{2}{x-1} dx - \int \frac{1}{x+2} dx = 2\ln|x-1| - \ln|x+2| + C = \ln(x-1)^2 - \ln|x+2| + C = \ln\frac{(x-1)^2}{|x+2|} + C$

e) Først polynomdivisjon: $\int \frac{x^3}{x^2-1} dx = \int x dx + \int \frac{x}{x^2-1} dx =$

Deretter delbrøk: $\int x dx + \frac{1}{2}\int \frac{1}{x+1} dx + \frac{1}{2}\int \frac{1}{x-1} dx =$

$\frac{x^2}{2} + \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C = \frac{1}{2}(x^2 + \ln|(x^2-1|) + C$

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c) Variabelskifte: $u = 1 + x^3 \Rightarrow u' = 3x^2 \Leftrightarrow dx = \frac{1}{3x^2} du$

$\int \frac{x^2}{\sqrt{1+x^3}} dx = \int \frac{x^2}{\sqrt{u}\,3x^2} du = \frac{1}{3}\int u^{-\frac{1}{2}} du = \frac{1}{3}\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C = \frac{1}{3}\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C =$
$\frac{2}{3}\sqrt{u} + C = \frac{2}{3}\sqrt{1+x^3} + C$

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d) Variabelskifte: $u = 2x^2 + 1 \Rightarrow u' = 4x \Leftrightarrow dx = \frac{du}{4x}$

$\int 2x\cos(2x^2+1) dx = \int 2x\cos(u)\frac{du}{4x} = \frac{1}{2}\int \cos u du = \frac{1}{2}\sin u + C = \frac{1}{2}\sin(2x^2+1) + C$

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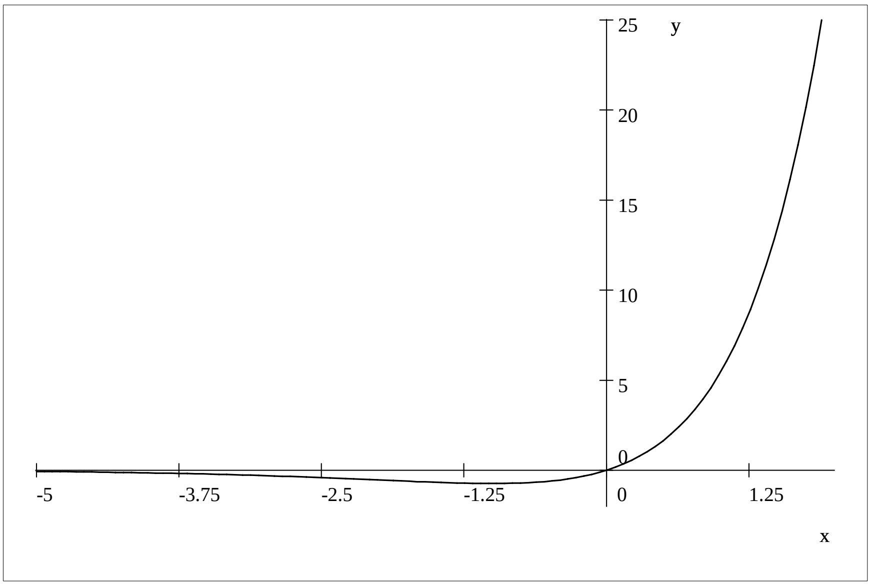
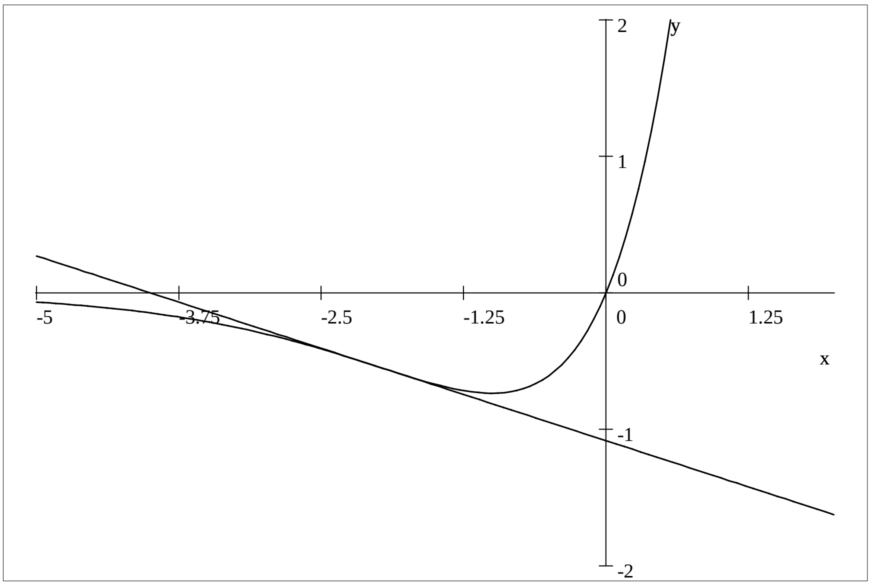
$f(x) = 2xe^x, \quad x \in [-4, 1]$



Produktregel: $f'(x) = 2e^x + 2xe^x = 2(1+x)e^x$
Enda en gang: $f''(x) = 2e^x + 2(1+x)e^x = 2(2+x)e^x$

Vendepunkt når $x = -2$: $(-2, 2(-2)e^{-2}) = (-2, -\frac{4}{e^2}) \approx (-2, -0.541)$

Vendetangent: $y - (-\frac{4}{e^2}) = f'(-2)(x - (-2)) \iff$
$y + \frac{4}{e^2} = -\frac{2}{e^2}(x+2) \iff y = -\frac{2}{e^2}x - \frac{8}{e^2}$
(Tilnærmet: $y = -0.271x - 1.08$)



Vendetangent skjærer x-aksen når: $-\frac{2}{e^2}x - \frac{8}{e^2} = 0 \iff x = -4$

Delvis: $\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C = (x-1)e^x + C$

$A = \left|\int_{-4}^{-2}(-\frac{2}{e^2}x - \frac{8}{e^2})dx\right| + \left|\int_{-2}^{0} 2xe^x dx\right| =$
$\left|-\frac{2}{e^2}\int_{-4}^{-2} x + 4dx\right| + \left|\int_{-2}^{0} 2xe^x dx\right| =$
$\frac{2}{e^2}\left|{}_{-4}^{-2}[\frac{x^2}{2} + 4x]\right| + 2|_{-2}^{0}[(x-1)e^x| =$

$\frac{2}{e^2}|2-8-(8-16)|+2|-1-(-3e^{-2})|=$
$\frac{2}{e^2}2+2|\frac{3}{e^2}-1|=\frac{4}{e^2}+2-\frac{6}{e^2}=2-\frac{2}{e^2}\approx 1.73$

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a) $u=\sin(2x)\Leftrightarrow u'=\cos(2x)2\Leftrightarrow dx=\frac{du}{2\cos(2x)}$
$\int\sin^2(2x)\cos(2x)dx=\int u^2\cos(2x)\frac{du}{2\cos(2x)}=\frac{1}{2}\int u^2du=\frac{1}{2}\frac{u^3}{3}+C=$
$\frac{1}{6}\sin^3(2x)+C$

$V=\pi\int_0^{\frac{\pi}{4}}\sin^2(2x)\cos(2x)dx=\frac{\pi}{6}{}_0^{\frac{\pi}{4}}[\sin^3(2x)]=\frac{\pi}{6}(1-0)=\frac{\pi}{6}$

b) $V=\pi\int_0^{2\pi}(\sin 2x+3)^2dx=\pi\int_0^{2\pi}(\sin^2(2x)+6\sin(2x)+9)dx=$
$\pi\int_0^{2\pi}(\frac{1}{2}-\frac{1}{2}\cos(4x)+6\sin(2x)+9)dx=$
$\pi_0^{2\pi}[\frac{1}{2}x-\frac{1}{8}\sin(4x)-3\cos(2x)+9x]=$
$\pi(\pi-3+18\pi-(-3\cdot 1))=19\pi^2\approx 188$

c) $(1-\tan x)^2=1-2\tan x+\tan^2 x$

$\int(1-2\tan x+\tan^2x)dx=x+2\ln|\cos(x)|+\tan x-x+C$

$V=\pi\int_0^{\frac{\pi}{4}}(1-2\tan x+\tan^2x)dx=\pi_0^{\frac{\pi}{4}}[x+2\ln|\cos(x)|+\tan x-x]=$
$\pi(\frac{\pi}{4}+2\ln\frac{\sqrt{2}}{2}+1-\frac{\pi}{4}-0)=\pi(2\ln\frac{\sqrt{2}}{2}+1)=\pi(2(\ln 2^{\frac{1}{2}}-\ln 2)+1)=$
$\pi(2(\frac{1}{2}\ln 2-\ln 2)+1)=\pi(1-\ln 2)\approx 0.964$

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$x=y^2\Leftrightarrow y=\pm\sqrt{x}$

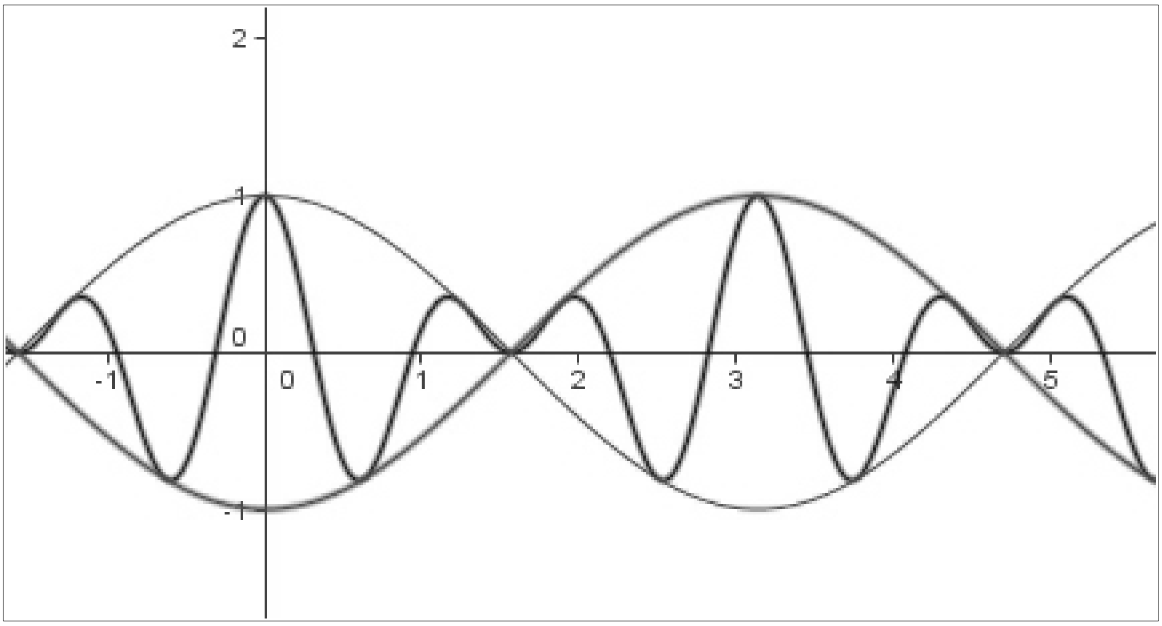
(Om vi dreier begge funksjonene 180° eller den positive 360° gir samme omdreiningslegeme!)

$V=\pi\int_0^{20}y^2dx=\pi\int_0^{20}xdx=\pi_0^{20}[\frac{x^2}{2}]=\pi(200-0)=200\pi\approx 628$

X5.4

Denne virker kanskje litt på siden, men har vært gitt til eksamen, hvis de ikke finner på noe bedre, kan de godt finne på å gi en variant av denne, feks. med $\sin(mx)\sin(nx)$ eller noe lignende!

$f(x)=\cos(x)\cos(5x)$
a, b)
$g(x)=\cos(x), h(x)=-\cos(x)$ gir i GeoGebra:



$\pm\cos(x)$ er altså den såkalte omhyllingskurven til $\cos(5x)$.

c) Ved å legge sammen ligningene får vi:

$$\cos(u-v)+\cos(u+v) = 2\cos u\cos v$$

eller:

$$\cos u\cos v = \tfrac{1}{2}(\cos(u-v)+\cos(u+v))$$

d) $u = x$ og $v = 5x$ gir:

$$\int \cos x\cos(5x)dx = \tfrac{1}{2}\int(\cos(x-5x)+\cos(x+5x))dx =$$
$$\tfrac{1}{2}\int(\cos(-4x)dx+\cos(6x))dx = \tfrac{1}{2}\int(\cos(4x)dx+\cos(6x))dx =$$
$$\tfrac{1}{2}\left(\tfrac{\sin(4x)}{4}+\tfrac{\sin(6x)}{6}\right)+C = \tfrac{1}{8}\sin 4x+\tfrac{1}{12}\sin 6x+C$$

e) $\cos mx \bullet \cos nx = \tfrac{1}{2}(\cos((m-n)x)+\cos((m+n)x)$

$$I = \int_0^{2\pi}\cos mx \bullet \sin nx\, dx = \tfrac{1}{2}\int_0^{2\pi}(\cos((m-n)x)+\cos((m+n)x)dx =$$
$$\tfrac{1}{2}\,{}_0^{2\pi}\left[\tfrac{\sin((m-n)x)}{m-n}+\tfrac{\sin((m+n)x)}{m+n}\right] = 0 \qquad (\text{Når } m \neq n)$$

f) $m = n$ gir: $\cos mx \bullet \cos nx = \tfrac{1}{2}(\cos(0)+\cos(2nx) = \tfrac{1}{2}(1+\cos(2nx))$

$$I = \tfrac{1}{2}\int_0^{2\pi}(1+\cos(2nx))dx = \tfrac{1}{2}\,{}_0^{2\pi}[x+\tfrac{1}{2n}\sin(2nx)] =$$
$$\tfrac{1}{2}(2\pi+0-(0+0) = \pi \qquad QED$$