Eksamen R2 - Høsten 2012

30.11.2012

Løsningsskisser

Del 1 - Uten hjelpemidler

Oppgave 1

a) Produktregel: $f(x) = e^x \cos x + e^x(-\sin x) = e^x(\cos x - \sin x)$

b) Kjerneregel: $g(x) = 5u^3, \qquad u = 1 + \sin x$
$g'(x) = 15u^2 \cos x = 15 \cos x(1 + \sin x)^2$

Oppgave 2

a) Variabelskifte: $u = 1 + \sin x$:
$\frac{du}{dx} = \cos x \qquad \Rightarrow dx = \frac{du}{\cos x}$
$\int \cos x(1 + \sin x)^3 dx = \int \cos x \, u^3 \frac{du}{\cos x} = \int u^3 du = \frac{u^4}{4} + C =$
$\frac{(1+\sin x)^4}{4} + C$

b) Delvis, med $v = \ln x$:
$\int x \ln x dx = \frac{x^2}{2} \ln x - \frac{1}{2} \int x^2 \frac{1}{x} dx = \frac{x^2}{2} \ln x - \frac{1}{2} \int x dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C =$
$\frac{x^2}{2}(\ln x - \frac{1}{2})$

$\int_1^e x \ln x dx =_1^e \left[\frac{x^2}{2}(\ln x - \frac{1}{2}) \right] = \frac{e^2}{2}(\ln e - \frac{1}{2}) - \frac{1^2}{2}(\ln 1 - \frac{1}{2}) =$
$\frac{e^2}{2}(1 - \frac{1}{2}) - \frac{1^2}{2}(0 - \frac{1}{2}) = \frac{e^2}{4} + \frac{1}{4} = \frac{1}{4}(e^2 + 1) \approx 2.10$

(Integraler kan kontrolleres med derivasjon.)

Oppgave 3

a) $\overrightarrow{AB} = [1, 0, 4], \qquad \overrightarrow{AC} = [2, 6, 2], \qquad \overrightarrow{BC} = [1, 6, -2]$

Hvis rettvinklet, må en av skalarproduktene være null:
$\overrightarrow{AB} \cdot \overrightarrow{AC} = 1 \cdot 2 + 0 \cdot 6 + 4 \cdot 2 = 10$

$\overrightarrow{AB} \cdot \overrightarrow{BC} = -7$

$\overrightarrow{BC} \cdot \overrightarrow{AC} = 34$

): Ingen rette vinkler i ΔABC

b) Parallellogram: $\overrightarrow{DC} = \overrightarrow{AB} = [1,0,4]$

$\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD} = \overrightarrow{OC} - \overrightarrow{DC} = [3,7,3] - [1,0,4] = [2,7,-1]$
): $D = (2,7,-1)$

(Kan kontrollere med $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD} = \overrightarrow{OA} + \overrightarrow{BC}$)

Oppgave 4

a) Karakteristisk ligning: $r^2 - 1 = 0 \Leftrightarrow (r-1)(r+1) = 0 \Leftrightarrow r = \pm 1$
$y = C_1 e^x + C_2 e^{-x}$

b) $y(0) = 5$: $\quad 5 = C_1 e^0 + C_2 e^0 \Leftrightarrow C_1 + C_2 = 5 \qquad (I)$

$y' = C_1 e^x - C_2 e^{-x}$

$y'(0) = -1$: $\quad -1 = C_1 e^0 - C_2 e^0 \Leftrightarrow C_1 - C_2 = -1 \qquad (II)$

$I + II$: $\quad 2C_1 = 4 \Leftrightarrow C_1 = 2$

Innsatt i I : $\quad 2 + C_2 = 5 \Leftrightarrow C_2 = 3$

Oppgave 5

Geometrisk rekke med $k = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = \frac{1}{3}$

Konvergent, da: $|k| < 1$

$S = \frac{a_1}{1-k} = \frac{1}{1-\frac{1}{3}} = \frac{3}{3-1} = \frac{3}{2}$

Oppgave 6

Amplitude: $\quad a = \frac{\text{maks-min}}{2} = \frac{8-2}{2} = 3$
Likevektslinje: $d = \frac{\text{maks+min}}{2} = \frac{8+2}{2} = 5$
Periode: Horisontal avstand mellom B og T er en kvart periode:
$\frac{P}{4} = 5 - 3 \Leftrightarrow P = 8$

$c = \frac{2\pi}{P} = \frac{2\pi}{8} = \frac{\pi}{4}$
Faseforskyvning: Ingen, da grafen til $f(x)$ skjærer likevektslinjen for $x = 0$.
Da har vi: $\varphi = 0$

): $\quad f(x) = 3\sin(\frac{\pi}{4}x) + 5$

Oppgave 7

a) Produktregel: $f'(x) = 2xe^{-x} + x^2 e^{-x}(-1) = e^{-x}(2x - x^2) = e^{-x}x(2-x)$

Ekstremalpunkter: $f'(x) = 0 \Leftrightarrow x(2-x) = 0 \Leftrightarrow x = 0 \vee x = 2$

Bunn-punkt: $(0, f(0)) = (0, 0)$
Topp-punkt: $(2, f(2)) = (2, 4e^{-2}) = (2, \frac{4}{e^2})$

b) Skisse.

Oppgave 8

$n = 1$:

$VS = 1 \cdot 4 = 4$

$HS = \frac{1(1+1)(1+5)}{3} = 4 \qquad OK!$

$n \to n + 1$:

Antar at formel gjelder for n, må vise at den da også gjelder for $n + 1$, det vil si at $P(n+1) = \frac{(n+1)((n+1)+1)((n+1)+5}{3} = \frac{(n+1)(n+2)(n+6)}{3}$

Rekursjon gir:

$$P(n+1) = P(n) + a_{n+1} = \frac{n(n+1)(n+5)}{3} + (n+1)((n+1)+3) =$$
$$\frac{n(n+1)(n+5)}{3} + (n+1)(n+4) = \frac{n(n+1)(n+5)+3(n+1)(n+4)}{3} =$$
$$\frac{n+1}{3}(n(n+5) + 3(n+4)) = \frac{n+1}{3}(n^2 + 5n + 3n + 12) =$$
$$\frac{n+1}{3}(n^2 + 8n + 12) = \frac{n+1}{3}(n+2)(n+6) \qquad OK!$$

Del 2 - Med hjelpemidler

Oppgave 1

a) Produktregel:

$$f'(x) = -8e^{-x}\sin 2x + 16e^{-x}\cos 2x = -8e^{-x}(\sin 2x - 2\cos 2x) =$$
$$-8e^{-x}\sqrt{1^2 + 2^2}\,\sin(2x + \varphi), \qquad \tan\varphi = \frac{-2}{1},\ \varphi \text{ i 4 KV}$$
$$-17.9e^{-x}\sin(2x - 1.11)$$

Ekstremalpunkter:

$$f'(x) = 0 \Leftrightarrow 2x - 1.11 = k\pi \Leftrightarrow x = 0.555 + k\frac{\pi}{2}$$

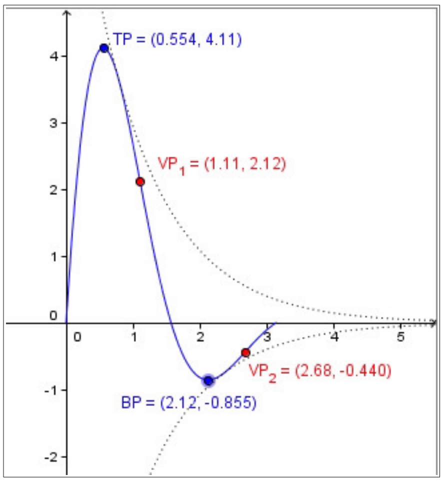
Topp-punkt: $TP = (0.555, f(0.555)) = (0.555, 4.11)$

Bunn-punkt: $BP = (0.555 + \frac{\pi}{2}, f(0.555 + \frac{\pi}{2})) = (2.12, -0.855)$

Vendepunkter:

$$f''(x) = -17.9(-e^{-x}\sin(2x - 1.11) + e^{-x}2\cos(2x - 1.11) =$$
$$17.9e^{-x}(\sin(2x - 1.11) - 2\cos(2x - 1.11) =$$
$$40.0e^{-x}(\sin(2x - 1.11 + (-1.11)) =$$
$$40.0e^{-x}\sin(2x - 2.22)$$
$$f''(x) = 0 \Leftrightarrow 2x - 2.22 = k\pi \Leftrightarrow x = 1.11 + k\frac{\pi}{2}$$

$VP_1 = (1.11, f(1.11)) = (1.11, 2.12)$
$VP_2 = (1.11 + \frac{\pi}{2}, f(1.11 + \frac{\pi}{2})) = (2.68, f(2.68)) = (2.68, -0.440)$



b) Formelen gir:

$$I(x) = 8\int e^{-x}\sin 2x\,dx = 8\frac{e^{-x}}{(-1)^2+2^2}(-\sin 2x - 2\cos 2x) + C =$$
$$-\frac{8}{5}e^{-x}(\sin 2x + 2\cos 2x) + C$$

Produktregel gir:

$$I'(x) = -\frac{8}{5}(-e^{-x}(\sin 2x + 2\cos 2x) + e^{-x}(2\cos 2x - 4\sin 2x)) =$$
$$\frac{8}{5}e^{-x}(2\cos 2x + \sin 2x) - \frac{8}{5}e^{-x}(2\cos 2x - 4\sin 2x) =$$
$$\frac{8}{5}e^{-x}(2\cos 2x + \sin 2x - 2\cos 2x + 4\sin 2x) =$$
$$\frac{8}{5}e^{-x}5\sin 2x = -8e^{-x}\sin 2x \qquad QED$$

c) $A = A_1 + A_2 = \int_0^{\frac{\pi}{2}} f(x)dx + \left|\int_{\frac{\pi}{2}}^{\pi} f(x)dx\right| =$

$$\left[-\frac{8}{5}e^{-x}(\sin 2x + 2\cos 2x)\right]_0^{\frac{\pi}{2}} + \left|\left[-\frac{8}{5}e^{-x}(\sin 2x + 2\cos 2x)\right]_{\frac{\pi}{2}}^{\pi}\right| =$$
$$-\frac{8}{5}e^{-\frac{\pi}{2}}(\sin\pi + 2\cos\pi) + \frac{8}{5}e^{0}(\sin 0 + 2\cos 0) +$$
$$\left|-\frac{8}{5}e^{-\pi}(\sin 2\pi + 2\cos 2\pi) + \frac{8}{5}e^{-\frac{\pi}{2}}(\sin\pi + 2\cos\pi\right| =$$
$$-\frac{8}{5}e^{-\frac{\pi}{2}}(0 + 2(-1)) + \frac{8}{5}(0+2) + \left|-\frac{8}{5}e^{-\pi}(0+2) + \frac{8}{5}e^{-\frac{\pi}{2}}(0 + 2(-1)\right|$$
$$\frac{16}{5}e^{-\frac{\pi}{2}} + \frac{16}{5} + \left|-\frac{16}{5}e^{-\pi} - \frac{16}{5}e^{-\frac{\pi}{2}}\right| =$$
$$\frac{16}{5}e^{-\frac{\pi}{2}} + \frac{16}{5} + \frac{16}{5}e^{-\pi} + \frac{16}{5}e^{-\frac{\pi}{2}} =$$
$$\frac{16}{5}e^{-\pi} + \frac{32}{5}e^{-\frac{1}{2}\pi} + \frac{16}{5} =$$
$$\frac{16}{5}(e^{-\frac{\pi}{2}} + 1)^2 \approx 4.6687$$

Oppgave 2

a) Separabel og lineær. Velger å separere:

$v \neq \frac{12}{1.15}$:

$$\int\frac{v'}{12-1.15v}dt = \int 1dt \Leftrightarrow \int\frac{1}{12-1.15v}dv = t + C_1 \Leftrightarrow$$
$$\frac{1}{-1.15}\ln|12 - 1.15v| = t + C1 \Leftrightarrow \ln|12 - 1.15v| = -1.15t + C_2 \Leftrightarrow$$
$$12 - 1.15v = C_3e^{-1.15t} \Leftrightarrow v = \frac{12}{1.15} - C_3e^{-1.15t} \Leftrightarrow$$
$$v = 10.4 - Ce^{-1.15t} \qquad \text{(Generell løsning.)}$$

($v = \frac{12}{1.15}$ er inkludert i løsningen over når $C = 0$)

$v(0) = 0$: $\quad 0 = 10.4 - Ce^0 \Leftrightarrow C = 10.4$

$v = 10.4 - 10.4e^{-1.15t}$ (Spesiell løsning.)

b) $s(t) = \int v dt = \int (10.4 - 10.4e^{-1.15t})dt = 10.4t - \frac{10.4}{-1.15}e^{-1.15t} + C =$
$10.4t + 9.04e^{-1.15t} + C$

$s(0) = 0$: $0 = 0 + 9.04e^0 + C \Leftrightarrow C = -9.04$

$s(t) = 10.4t + 9.04e^{-1.15t} - 9.04$ [m]

c) $s(t) = 100 \Leftrightarrow 100 = 10.4t + 9.04e^{-1.15t} - 9.04$

Må løses numerisk, for eksempel som skjæringspunkt mellom høyre og venstre side av ligningen, enten på lommeregner eller i GeoGeBra. Skriv hva du gjør!

Sprinteren vil bruke ca. 10.5 sekunder på 100 m ifølge oppgavens modell.

Oppgave 3

a)
1) Vektorene står normalt på hverandre. (Eller en eller begge er nullvektor.)

2) Vektorene er parallelle. (Eller en eller begge er nullvektor.)

3) Vektorene ligger i samme plan. (Eller en eller to eller tre er nullvektor.)

b) $(\vec{a} \times \vec{b})^2 + \vec{a} \cdot \vec{b}^2 = \left|\vec{a} \times \vec{b}\right|^2 + \left|\vec{a} \cdot \vec{b}\right|^2 =$
$(\left|\vec{a}\right|\left|\vec{b}\right| \sin\alpha)^2 + (\left|\vec{a}\right|\left|\vec{b}\right| \cos\alpha)^2 =$
$\left|\vec{a}\right|^2\left|\vec{b}\right|^2(\sin^2\alpha + \cos^2\alpha) = \left|\vec{a}\right|^2\left|\vec{b}\right|^2$ *QED*

c) , b) gir oss:
$\left|\vec{a} \times \vec{b}\right| = \sqrt{\left|\vec{a}\right|^2\left|\vec{b}\right|^2 - (\vec{a} \cdot \vec{b})^2}$

Arealformelen for en trekant, $A = \frac{ab\sin\alpha}{2}$, kan skrives som
$A = \frac{\left|\vec{a}\times\vec{b}\right|}{2} = \frac{\sqrt{\left|\vec{a}\right|^2\left|\vec{b}\right|^2-(\vec{a}\cdot\vec{b})^2}}{2}$ *QED*

d) $\overrightarrow{AB} = [1,2,3]$, $\left|\overrightarrow{AB}\right|^2 = 1^2 + 2^2 + 3^2 = 14$
$\overrightarrow{AC} = [2,0,4]$, $\left|\overrightarrow{AC}\right|^2 = 2^2 + 0^2 + 4^2 = 20$
$\overrightarrow{AB} \cdot \overrightarrow{AC} = 1 \cdot 2 + 2 \cdot 0 + 3 \cdot 4 = 14$

$A_{ABC} = \frac{\sqrt{\left|\vec{a}\right|^2\left|\vec{b}\right|^2-(\vec{a}\cdot\vec{b})^2}}{2} = \frac{\sqrt{14\cdot20-14^2}}{2} = \sqrt{21} \approx 4.58$

Oppgave 4

a) Aritmetisk rekke med $d = 2$:

$S_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}(1 + 2n - 1) = n^2$

Nødvendig antall ledd:

$S_n = 1600 \Leftrightarrow n^2 = 1600 \Leftrightarrow n = 400 \quad (n = -400$ forkastes.)

b) Geometrisk rekke med $k = \frac{a_2}{a_1} = \frac{a3}{a_2} = \frac{a_4}{a_3} = \frac{1}{4}$

$S = \frac{a_1}{1-k} = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$

Oppgave 5

a) $(\frac{y}{b})^2 = 1 - (\frac{x}{a})^2 \Leftrightarrow y^2 = b^2(1 - (\frac{x}{a})^2) \Leftrightarrow y^2 = b^2 - \frac{b^2}{a^2}x^2 \qquad QED$

b) Volum:

$$V = \pi \int_{-a}^{a} y^2 dx = \pi \int_{-a}^{a} (b^2 - \frac{b^2}{a^2}x^2) dx =$$
$$\pi \,{}_{-a}^{\;a}\Big[b^2 x - \frac{b^2}{a^2}\frac{x^3}{3} \Big] = \pi(b^2 a - \frac{b^2 a}{3} - (-b^2 a + \frac{b^2 a}{3})) =$$
$$\pi(2b^2 a - \frac{2}{3}b^2 a) = \frac{4}{3}\pi a b^2 \qquad QED$$

(Legg merke til at $a = b = r$ gir formelen for volumet av en kule.)

Oppgave 6

a)

$f'(x) = (x-2)^{\frac{1}{2}\prime} = \frac{1}{2}(x-2)^{-\frac{1}{2}} = \frac{1}{2\sqrt{x-2}}$

Ett-punkts-formelen: $y - f(a) = f'(a)(x - a)$

gir da:

$$y - \sqrt{a-2} = \frac{1}{2\sqrt{a-2}}(x - a) \Leftrightarrow$$
$$y = \frac{1}{2\sqrt{a-2}}x - \frac{a}{2\sqrt{a-2}} + \sqrt{a-2} \Leftrightarrow$$
$$y = \frac{1}{2\sqrt{a-2}}x + (\frac{2(a-2)}{2\sqrt{a-2}} - \frac{a}{2\sqrt{a-2}}) \Leftrightarrow$$
$$y = \frac{1}{2\sqrt{a-2}}x + \frac{a-4}{2\sqrt{a-2}} \qquad QED$$

A : $\quad \frac{1}{2\sqrt{a-2}}x + \frac{a-4}{2\sqrt{a-2}} = 0 \Leftrightarrow x = 4 - a$

): $\quad A = (4 - a, 0)$

B : $\quad f(x) = 0 \Leftrightarrow \sqrt{x-2} = 0 \Leftrightarrow x = 2$

): $\quad B = (2, 0)$

$C = (a, 0)$

b) $A_f = \int_2^a \sqrt{x-2}\, dx = \left[\frac{(x-2)^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right]_2^a = \left[\frac{2}{3}(x-2)^{\frac{3}{2}} \right]_2^a = \frac{2}{3}(a-2)^{\frac{3}{2}} - 0 =$

$\frac{2}{3}(a-2)^{\frac{3}{2}} \qquad QED$

$$A_{ACP} = \frac{gh}{2} = \frac{AC \cdot CP}{2} = \frac{(a-(4-a))\sqrt{a-2}}{2} = \frac{1}{2}\sqrt{a-2}\,(2a-4) =$$
$$\sqrt{a-2}\,(a-2) = (a-2)^{\frac{3}{2}}$$

c) $\frac{A_2}{A_1} = \frac{A_f}{A_{ACP} - A_f} = \frac{\frac{2}{3}(a-2)^{\frac{3}{2}}}{(a-2)^{\frac{3}{2}} - \frac{2}{3}(a-2)^{\frac{3}{2}}} = \frac{\frac{2}{3}(a-2)^{\frac{3}{2}}}{(a-2)^{\frac{3}{2}}(1-\frac{2}{3})} =$

$\frac{\frac{2}{3}}{1-\frac{2}{3}} = \frac{2}{3-2} = 2 \qquad QED$